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A FEASIBILITY CRITERION FOR STAIRCASE TRANSPORTATION PROBLEMS AND AN APPLICATION TO A SCHEDULING PROBLEM

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SUMMARY

A feasibility criterion for transportation problems in which certain variables are inadmissible is shown to yield a simple feasibility test for such problems when the admissible set has the form of a staircase. A simple rule is then presented for singling out a feasible solution for staircase problems. As an application of these results, it is shown that a particular case of the problem of minimizing the number of carriers to meet a fixed schedule can be solved explicitly by an appropriate interpretation of the staircase rule.

A FEASIBILITY CRITERION FOR STAIRCASE TRANSPORTATION PROBLEMS AND AN APPLICATION TO A SCHEDULING PROBLEM

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INTRODUCTION

In this note we apply a general feasibility criterion for transportation problems in which some variables are inadmissible to obtain a simple feasibility test for such problems when the admissible set has a certain form, dubbed staircase (§2). As a by-product, an easily applied rule is presented for picking out a feasible solution to the latter problem when one exists.

As an application of these results, it is shown that a particular case of the problem of minimizing the number of carriers to meet a fixed schedule [1] can be solved directly by an appropriate interpretation of the procedure for selecting a feasible solution. Thus, for example, the following problem has an easy solution. Suppose given a network which is simply a chain, and assume that pickup points $P_1, \ldots, P_p$ are at one end of the chain, discharge points $D_1, \ldots, D_q$ at the other end, and that each arc of the chain has a given traversal time. If, for each origin-destination pair P_r, D_s, there is given a fixed schedule of times t^1_{rs}, t^2_{rs}, ... at which a loaded carrier is to leave P_r bound for D_s (there to be unloaded and reassigned to pick up any other load that it can reach in time), how many carriers are required to meet the schedule?

1. FEASIBILITY AND REDUCTION CRITERIA

Various necessary and sufficient conditions for feasibility of the transportation constraints

$$(1)\qquad \begin{aligned} &\sum_{j=1}^{n} x_{ij} = a_i && i = 1, \ldots, m \\ &\sum_{i=1}^{m} x_{ij} \leq b_j && j = 1, \ldots, n \\ &x_{ij} \geq 0 \end{aligned}$$

where certain of the variables are inadmissible (i.e., fixed at zero), say

$$(2)\qquad x_{ij} = 0 \text{ for a given set } \mathcal{J} \text{ of pairs } (i, j),$$

are easily deduced from either the max-flow min-cut theorem [2] or from Gale's feasibility theorem [3] for network flows. If the set $\mathcal{J}$ is not specialized in any way, these conditions (see Theorem 1 below) are impracticable to apply unless m or n is small. However, for the case we will be interested in, the conditions become very simple, as will be seen.

To state Theorem 1, we require some definitions. For each row $i = 1, \ldots, m$ of the m by n transportation array, define the <u>span</u> of I (denoted $\sigma(I)$) to be all of those columns, say $j_1, \ldots, j_k$, for which $ij_1, \ldots, ij_k$ belong to the admissible set $\mathcal{A} = \bar{\mathcal{J}}$; extend this definition to subsets $I = \{i_1, \ldots, i_\ell\}$ of the rows by

$$\sigma(I) = \sigma(i_1) \cup \ldots \cup \sigma(i_\ell).$$

If I and J are subsets of the rows and columns, respectively, having the property that $ij \in \mathcal{O}\!\mathcal{l}$ implies either $i \in I$ or $j \in J$, the set $(I;J)$ is a *covering* of $\mathcal{O}\!\mathcal{l}$ [4]. In particular, the set of all rows (or the set of all columns) is always a covering. A covering may be termed *proper* if no proper subset covers $\mathcal{O}\!\mathcal{l}$.

Thus, for example, in the transportation array of Fig. 1, where inadmissible cells are shown crossed out, $\sigma(\{1, 2\}) = \{2, 3, 5, 6, 7\}$ and $(1, 3, 4;\ 2, 5, 6)$ is a proper covering of $\mathcal{O}\!\mathcal{l}$.

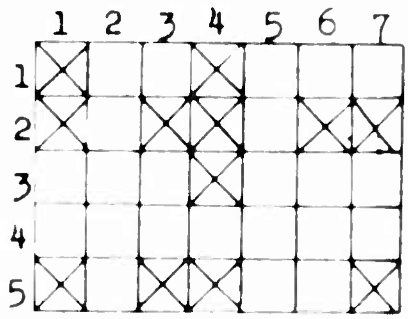

Fig. 1

Theorem 1. Suppose $a_i \geq 0$, $b_j \geq 0$, *and let* $A = \sum_{i=1}^{m} a_i$.
Then the following statements are equivalent:

(i) *the constraints* (1) *and* (2) *are feasible;*

(ii) *for every set* I *of rows,* $\sum_{i \in I} a_i \leq \sum_{j \in \sigma(I)} b_j$;

(iii) *for every (proper) covering* $(I;\ J)$, $\sum_{i \in I} a_i + \sum_{j \in J} b_j \geq A$.

Proof. Assume (i). Since the rectangle $I \times \overline{\sigma(I)}$ is contained in $\mathcal{J}$, we have

$$\sum_{i \in I} a_i = \sum_{i \in I} \sum_{j=1}^{n} x_{ij} = \sum_{i \in I} \sum_{j \in \sigma(I)} x_{ij} \leq \sum_{i=1}^{m} \sum_{j \in \sigma(I)} x_{ij} \leq \sum_{j \in \sigma(I)} b_j ,$$

verifying (ii). To prove that (ii) implies (iii), notice that if $(I;J)$ is a covering, then $\sigma(\overline{I})$ is contained in J. Hence

$$A - \sum_{i \in I} a_i = \sum_{i \in \overline{I}} a_i \leq \sum_{j \in \sigma(\overline{I})} b_j \leq \sum_{j \in J} b_j .$$

The proof will be completed by showing that (iii) implies (i). To do this, we shall use the max-flow min-cut theorem applied to a representing network N for the constraints (1) and (2).

Let N consist of nodes $s, p_1, \ldots, p_m, q_1, \ldots, q_n, t$ and directed arcs

sp_i	with capacity $c(s, p_i) = a_i$		$(i = 1, \ldots, m)$
q_jt	with capacity $c(q_j, t) = b_j$		$(j = 1, \ldots, n)$
p_iq_j	with capacity $c(p_i, q_j) = \infty$		$(ij \in \mathcal{A})$.

The constraints (1) and (2) are feasible if and only if the value of a maximal flow from s to t is A. Hence it suffices to show that every cut capacity exceeds A. A cut in N is a partition of the nodes into two sets X, $\overline{X}$ with $s \in X$, $t \in \overline{X}$; the capacity $c(X, \overline{X})$ of the cut is the sum of the capacities of all arcs leading from X to $\overline{X}$. Thus

$$c(X, \overline{X}) = \sum_{p_i \in \overline{X}} c(s, p_i) + \sum_{\substack{p_i \in X \\ q_j \in \overline{X}}} c(p_i, q_j) + \sum_{q_j \in X} c(q_j, t) ,$$

and $c(X, \overline{X})$ is infinite unless the set of arcs $p_i q_j$ with $p_i \in X$, $q_j \in \overline{X}$ is vacuous. If this set is vacuous, however, we see from the definition of N that $ij \in \mathfrak{A}$ implies that either $p_i \in \overline{X}$ or $q_j \in X$. Hence (I; J), where

$$I = \{i \mid p_i \in \overline{X}\}$$

$$J = \{j \mid q_j \in X\} ,$$

is a covering of $\mathfrak{A}$, and consequently (iii) implies

$$c(X, \overline{X}) = \sum_{i \in I} a_i + \sum_{j \in J} b_j \geq A.$$

Thus all cut capacities exceed A, proving (i). Obviously (iii) may be restricted to proper coverings, if desired.

The proof of the last implication of Theorem 1 establishes the following corollary of the max-flow min-cut theorem.

<u>Theorem 2.</u> <u>The maximum of</u> $\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}$ <u>subject to the constraints</u> (2) <u>and</u>

$$\sum_{j=1}^{n} x_{ij} \leq a_i$$

(3)

$$\sum_{i=1}^{m} x_{ij} \leq b_j$$

$$x_{ij} \geq 0$$

<u>is equal to the minimum of</u> $\sum_{i\epsilon I} a_i + \sum_{j\epsilon J} b_j$ <u>taken over all proper coverings</u> (I; J).

With respect to a given covering (I; J) of $\mathcal{O}\!\mathcal{L}$, we shall say that a cell $ij\epsilon\mathcal{O}\!\mathcal{L}$ is <u>singly</u> covered or <u>doubly</u> covered according as just one of the relations $i\epsilon I$, $j\epsilon J$ holds or both do. The next theorem provides a reduction criterion for the problem of finding a feasible solution to (1) and (2) (or what is the same thing, for maximizing the total sum of variables subject to (2) and (3)).

<u>Theorem 3</u>. <u>Suppose</u> $ij\epsilon\mathcal{O}\!\mathcal{L}$ <u>is singly covered in every proper covering of</u> $\mathcal{O}\!\mathcal{L}$. <u>Then setting</u> $x_{ij} = \min(a_i, b_j)$ <u>yields a new transportation problem whose feasibility is equivalent to that of the original problem</u>.

<u>Proof</u>. The new problem is obtained from the old by reducing a_i and b_j by $\min(a_i, b_j)$. Clearly if the new problem is feasible, so is the old. On the other hand, if the original problem is feasible, then (iii) of Theorem 1 holds for all proper coverings (I; J) of $\mathcal{O}\!\mathcal{L}$. Since ij is singly covered in each proper covering, both sides of every inequality $\sum_{i\epsilon I} a_i + \sum_{j\epsilon J} b_j \geq A$ are reduced by the same ammount in the new problem, and consequently (iii) remains valid.

Notice, conversely, that any cell ij which is doubly covered in some proper covering may not be a candidate for immediate evaluation in checking the feasibility of a given problem, or in solving the related maximum problem. Thus the futility of attempting to devise direct methods of solution which will work for all such problems is apparent. However, for the particular

class of problems dealt with in the next section, Theorem 3 does provide such a method for the determination of a feasible solution if one exists.

2. STAIRCASE FEASIBILITY PROBLEM

The transportation constraints (1) and (2) will be said to be in staircase form (with K steps) if they may be written as

$$\sum_{j=1}^{s_1} x_{ij} = a_i \quad (i = 1,\ldots,r_1) \qquad \sum_{i=1}^{r_K} x_{ij} \leq b_j \quad (j=1,\ldots,s_1)$$

$$\sum_{j=1}^{s_2} x_{ij} = a_i \quad (i = r_1+1,\ldots,r_2) \qquad \sum_{i=r_1+1}^{r_K} x_{ij} \leq b_j \quad (j=s_1+1,\ldots,s_2)$$

$$- - - -$$

$$\sum_{1}^{s_K} x_{ij} = a_i \quad (i = r_{K-1}+1,\ldots,r_K) \qquad \sum_{i=r_{K-1}+1}^{r_K} x_{ij} \leq b_j \quad (j=s_{K-1}+1,\ldots,s_K)$$

where $1 \leq r_1 < \cdots < r_K = m$, $1 \leq s_1 < s_2 < \cdots < s_K = n$. Thus the constraints (1) and (2) may be pictured as shown in the example of Fig. 2, where $r_1 = 1$, $r_2 = 2$, $r_3 = 3$, $r_4 = 5$, and $s_1 = 1$, $s_2 = 3$, $s_3 = 4$, $s_4 = 8$.

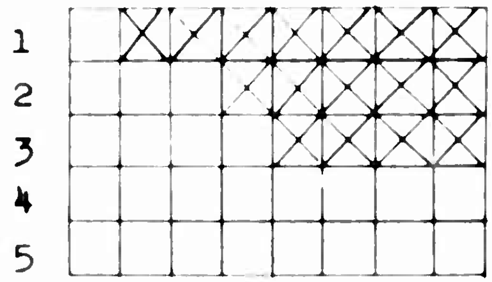

Fig. 2

To discover whether (1) and (2) may be put in staircase form, simply arrange the rows in order of increasing number of admissible cells, the columns in order of decreasing number of admissible cells. If the constraints are not staircase after this arrangement, no rearrangement will work.

Define, for $k = 1, \ldots, K$,

$$(5) \qquad \begin{aligned} A_k &= \sum_{i=r_{k-1}+1}^{r_k} a_i \qquad (r_o = 0) \\ B_k &= \sum_{j=s_{k-1}+1}^{s_k} b_j \qquad (s_o = 0) . \end{aligned}$$

The following theorem is an easy consequence of Theorem 1.

<u>Theorem 4. Suppose</u> $a_i \geq 0$, $b_j \geq 0$. <u>If the constraints (1) and (2) are staircase with</u> K <u>steps, the problem is feasible if and only if</u>

$$\sum_{i=1}^{k} A_i \leq \sum_{i=1}^{k} B_i, \qquad k = 1, \ldots, K.$$

Proof. Either of the conditions of Theorem 1 leads directly to the condition of Theorem 4. Suppose we apply (iii). Notice that the only proper coverings of $\mathfrak{A}$ are those $(I; J)$ where $I = \{r_{k-1}+1, \ldots, r_K = m\}$, $J = \{1, \ldots, s_{k-1}\}$, $k = 1, \ldots, K+1$. Thus (iii) is equivalent to the $K+1$ inequalities

$$
(6)\qquad
\begin{aligned}
&\sum_{i=1}^{K} A_i && \geq A \\
&\sum_{i=2}^{K} A_i + B_1 && \geq A \\
&\sum_{i=3}^{K} A_i + \sum_{i=1}^{2} B_i && \geq A \\
&A_K + \sum_{i=1}^{K-1} B_i && \geq A \\
&\sum_{i=1}^{K} B_i && \geq A,
\end{aligned}
$$

the first of which is an equality. Writing $A = \sum_{i=1}^{K} A_i$ in the remaining inequalities gives the condition of Theorem 4.

Using Theorem 3, a simple rule for picking a feasible solution (or solving the related maximum problem) may be deduced for staircase problems. Let $I_k (J_k)$ be the set of indices $i = r_{k-1}+1, \ldots, r_k$ $(j = s_{k-1}+1, \ldots, s_k)$, for $k = 1, \ldots, K$. Notice that the rectangles $I_k \times J_k$ are singly covered in all proper coverings. Thus we may select any cell ij from one of these rectangles and set $x_{ij} = \min(a_i, b_j)$, thereby deleting

a row or column (or both), and leaving a smaller staircase problem. The process may then be repeated. After at most $m + n - 1$ steps, a feasible solution has been constructed.

Moreover, since all other admissible cells are doubly covered in some proper covering, this is the only safe procedure to use in attempting to find a solution by the reduction process of Theorem 3.

We also point out the obvious fact that if the row and columns sums a_i and b_j are integers, the staircase rule yields an integral solution to the feasibility problem or its related maximum problem.

3. A CARRIER SCHEDULING PROBLEM

It has been shown in [1] that the problem of minimizing the number of carriers required to meet a fixed schedule can be reduced to the transportation problem of maximizing the total sum of variables subject to constraints of the form (2) and (3). Under certain circumstances, the admissible set for this transportation problem is staircase, thereby permitting a trivial solution.

The general problem may be described as follows. A rectangular array of spaces is given, one row for each pickup point P_r and one column for each discharge point D_s. In space r, s there is a finite sequence $t_{r,s}$ of positive integers representing the times at which a carrier is to load at pickup point P_r for delivery to destination D_s. In addition, two arrays of positive integers a_{rs} and b_{rs} are given, where a_{rs} is

the loading-traveling time from P_r to D_s and b_{rs} the unloading-traveling time from D_s to P_r. The problem is to meet the fixed schedule given by the array of sequences with a minimum number of carriers.

A particular version of the problem which can be solved by the staircase rule is obtained by specializing the matrix b_{rs} of reassignment times to have the form

$$\left[b_{rs}\right] = \begin{bmatrix} b_1 + c_1, & b_1 + c_2, & \ldots, & b_1 + c_q \\ b_2 + c_1, & b_2 + c_2, & \ldots, & b_2 + c_q \\ & & \cdots\cdots & \\ b_p + c_1, & b_p + c_2, & \ldots, & b_p + c_q \end{bmatrix}$$

We then define

$a(\alpha, r)$: number of carriers required at P_r at time $\alpha + b_r$,

$b(\beta, s)$: number of carriers available at D_s at time $\beta - c_s$.

Thus $a(\alpha, r)$ may be computed from the given table of sequences and $b(\beta, s)$ from this table and the matrix a_{rs} of loading-traveling times. The range of α, β may be taken to be 1, ..., K, where K is chosen sufficiently large to include all positive $a(\alpha, r)$, $b(\beta, s)$.

For any given routing of carriers to meet the schedule, let $x(\alpha, r; \beta, s)$ denote the number of carriers located at D_s at time $\beta - c_s$ which are reassigned to pick up loads at P_r at time $\alpha + b_r$. Thus the constraints

$$\sum_{\alpha, r} x(\alpha, r; \beta, s) \leq b(\beta, s),$$

(7)

$$\sum_{\beta, s} x(\alpha, r; \beta, s) \leq a(\alpha, r),$$

(8)
$$x(\alpha, r; B, s) \geq 0,$$
$$x(\alpha, r; \beta, s) = 0 \text{ if } (\beta - c_s) + (b_r + c_s) > \alpha + b_r,$$

are satisfied for all routings. Conversely, any integral solution to the constraints (7) and (8) may be used to construct routings for M carriers, where M is the difference between the total number of entries in the table of sequences and the total number of reassignments $\sum_{\alpha, r, \beta, s} x(\alpha, r; \beta, s)$. Hence an integral solution to (7) and (8) which maximizes

(9) $$\sum_{\alpha, r, \beta, s} x(\alpha, r; \beta, s)$$

minimizes the number of carriers required.

If we order the rows (α, r) and columns (β, s) of the transportation constraints (7) lexicographically, say, we see from (8) that the admissible set, defined by $\beta \leq \alpha$, is staircase. A particular interpretation of the staircase rule then leads to the following routing doctrine: If an empty is available at D_s at time α and if, for any r, there is a load scheduled to leave P_r at time $\alpha + b_{rs}$, assign the empty to P_r; otherwise, look next for loads leaving the pickup points at

times $\alpha + 1 + b_{rs}$, and so on. One can apply this rule first to construct a routing for one carrier, then (in the reduced table of sequences) for another carrier, etc. The resulting routing is one using the minimal number of carriers.

Also an explicit formula for the minimal number of carriers required can be written down. Let T be the total number of entries in the given table of sequences, and define

$$(10)\qquad \begin{aligned} A(\alpha) &= \sum_r a(\alpha, r) \qquad\qquad \alpha = 1, \ldots, K \\ B(\alpha) &= \sum_s b(\alpha, s) \ . \end{aligned}$$

Then (see Theorem 2 and inequalities (6)) the minimal number of carriers is given by

$$(11)\qquad M_{min} = T - \min_{1 \leq k \leq K+1} \left[\sum_{\alpha=k}^{K} A(\alpha) + \sum_{\alpha=1}^{k-1} B(\alpha) \right].$$

We conclude with a numerical example. Suppose there are two pickup and two discharge points with fixed schedules given by the table

	D_1	D_2
P_1	1, 4, 6, 7, 10, 12, 13	9, 15
P_2	3, 5, 6, 9, 10, 12, 15	7, 10, 13, 15

and let

$$\left[a_{rs}\right] = \left[b_{rs}\right] = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} .$$

Thus we may take $b_1 = 2$, $b_2 = 1$, $c_1 = 0$, $c_2 = 1$.

The functions $A(\alpha)$ and $B(\alpha)$ are tabulated below:

α	$A(\alpha)$	$B(\alpha)$
1	0	0
2	2	0
3	0	1
4	2	1
5	2	0
6	1	2
7	1	1
8	2	1
9	2	1
10	1	2
11	2	1
12	1	1
13	1	3
14	2	1
15	0	1
16	0	2
17	0	0
18	0	1
19	0	1

The minimum of the bracketed expression in (11), which occurs for $k = 10$, is 14, and thus $M_{min} = 20 - 14 = 6$. Individual routings for the six carriers, obtained using the routing rule previously described, follow:

Carrier

time	1	2	3	4	5	6
1	P_1					
2						
3	D_1	P_2				
4		D_1	P_1			
5		P_2				
6	P_1	D_1	D_1	P_2		
7		P_2		D_1	P_1	
8	D_1				D_1	
9		D_2	P_1	P_2		
10	P_1			D_1	P_2	P_2
11						D_1
12	D_1	P_1	D_2		D_2	P_2
13	P_2			P_1		D_1
14		D_1				
15	D_2	P_2	P_1	D_1	P_2	
16					D_1	
17		D_2				
18			D_2			

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